Error Rate Analysis for Coded Multicarrier Systems over Quasi-static Fading Channels

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Outline

1. Overview of Concepts

2. Performance Analysis of Coded OFDM Systems
    - Error Vectors
    - Pairwise Error Probability
    - Per-realization BER
    - Direct Average BER

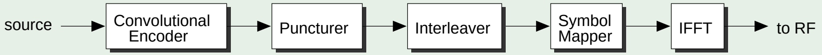
3. Performance Results for Multiband OFDM

Overview of Concepts

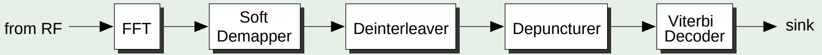
Orthogonal Frequency Division Multiplexing (OFDM)

- Many orthogonal subchannels via IFFT & FFT
- (Punctured) convolutional codes for error protection
- Robust in frequency-selective fading

Transmitter



Receiver



Overview of Concepts

Quasi-Static Fading Channel

- Fading conditions constant for “relatively long” period of time
- Here: “relatively long” $=$ (at least) one packet
- Can only code over one channel realization
- Channel is also frequency-selective

For a coded packet, each subcarrier has a fixed channel gain

Analysis: System Model

Frequency-domain System Model

$$\mathbf{r} = \mathbf{H}\mathbf{x} + \mathbf{J} + \mathbf{n}$$

where

- $\mathbf{x}$ are the transmitted symbols
- $\mathbf{H} = \mathrm{diag}(\mathbf{h})$ is a (diagonal) matrix of channel gains
- $\mathbf{J}$ is the (frequency-domain) interference signal
- $\mathbf{n}$ are AWGN noise variables

Interference Model

$$i(t) = \sum_{k=1}^{N_i} \alpha_k e^{j(2\pi f_k t + \phi_k)}$$

$$\mathbf{J} = \mathrm{DFT}\left(\begin{bmatrix} i(0) & i(T) & i(2T) & \ldots & i((N-1)T) \end{bmatrix}\right)$$

Analysis: Performance Measures

We will consider two different performance measures

Average Bit Error Rate (BER)

Average BER of a (large) number of different channel realizations

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Outage BER

- if we assume $X\%$ of channels are bad ("in outage"), what is the worst-case performance of the remaining $(100 - X)\%$ of the channels?
- (we will consider $X = 10\%$ outage probability)

Outage BER is a more relevant measure for quasi-static channels...
BUT, it is hard to determine.

Analysis: BER for Quasi-static Channels

Average BER for Fast Fading Channel

- Get pairwise error probability $P_2(d|h)$ for given h
- Integrate over pdf of h to get $P_2(d)$, then apply union bound

$$\mathrm{BER} \leq \frac{1}{k} \sum_{d=d_{\mathrm{free}}}^{\infty} \beta_d P_2(d)$$

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Quasi-static Channel

Each coded block transmitted over one channel realization

- Only "sees" a limited number of channel gains
- (and, correlation between bits depends on starting position)

CANNOT simply integrate over pdf of fading distribution!

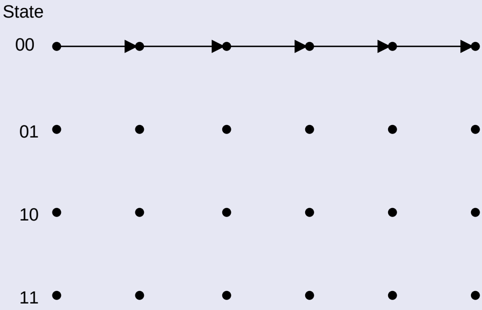
Question

Can we predict the BER of a given realization?

Analysis: Error Vectors

Trellis representation of $R_c = 1/2$ (7,5) convolutional code

An example error path



Input: $\{\ ,\ ,\ ,\ \}$

Output $\mathbf{e}_1$: $\{\ ,\ ,\ ,\ ,\ ,\ ,\ ,\ \}$

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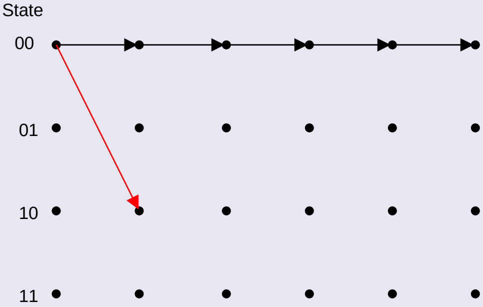

Input: $\{1, \ , \ , \ \}$

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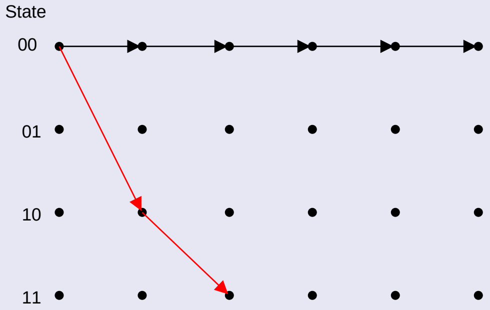

Input: $\{1, 1, \ , \ \}$

Output $\mathbf{e}_1$: $\{1, 1, 0, 1, \ , \ , \ , \ \}$

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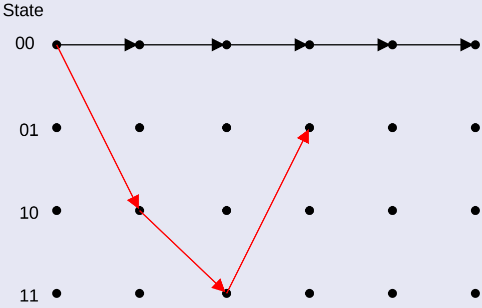

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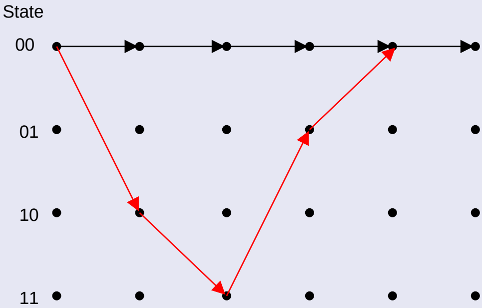

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- Call this set $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_L\}$

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- Make the set of all error vectors $\mathbf{e}_\ell$ of less than a certain weight (L of them in total)
- Call this set $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_L\}$
- l_ℓ is the length of $\mathbf{e}_\ell$
- a_ℓ is the number of information bit errors associated with $\mathbf{e}_\ell$

$\mathcal{E}$ contains all the "most likely" error events (the ones with shortest length)

Analysis: PEP

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- For each error vector $\mathbf{e}_\ell$, we form the "full" error codeword (length L_c)

$$\mathbf{q}_{i,\ell} = [\underbrace{0 \;\; 0 \; \dots \; 0}_{i-1} \; \underbrace{\mathbf{e}_\ell}_{l_\ell} \; \underbrace{0 \;\; 0 \; \dots \; 0}_{L_c-l_\ell-i+1}]^T$$

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- Now, given $\mathbf{q}_{i,\ell}$, the competing codeword is given by

$$\mathbf{v}_{i,\ell} = \mathbf{c} \oplus \mathbf{q}_{i,\ell}$$

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- We can map the two codewords into corresponding modulated symbols:
- true codeword $\mathbf{c} \longrightarrow \mathbf{x}$
- competing codeword $\mathbf{v}_{i,\ell} \longrightarrow \mathbf{z}_{i,\ell}$
- the PEP for the ℓth error vector starting in the ith position, i.e., the probability that $\mathbf{v}_{i,\ell}$ is detected given that $\mathbf{c}$ was transmitted, is given by

$$\mathrm{PEP}_{i,\ell}(\mathbf{H}, \mathbf{J}) = \Pr\left\{ ||\mathbf{r} - \mathbf{H}\mathbf{x}||^2 > ||\mathbf{r} - \mathbf{H}\mathbf{z}_{i,\ell}||^2 \;\middle|\; \mathbf{H}, \mathbf{J} \right\}$$

Analysis: Per-realization BER

For given $\mathbf{H}, \mathbf{J}, i, \ell$

PEP

$$\mathrm{PEP}_{i,\ell}(\mathbf{H},\mathbf{J}) = Q\left(\frac{\frac{1}{2}||\mathbf{H}(\mathbf{x}-\mathbf{z}_{i,\ell})||^2 + \mathrm{Re}\left\{\mathbf{J}^H\mathbf{H}(\mathbf{x}-\mathbf{z}_{i,\ell})\right\}}{\sqrt{\frac{1}{2}\mathcal{N}_0||\mathbf{H}(\mathbf{x}-\mathbf{z}_{i,\ell})||^2}}\right)$$

PEP if $\mathbf{J} = \mathbf{0}$

$$\mathrm{PEP}_{i,\ell}(\mathbf{H},\mathbf{J}) = Q\left(\sqrt{\frac{||\mathbf{H}(\mathbf{x}-\mathbf{z}_{i,\ell})||^2}{2\mathcal{N}_0}}\right)$$

Analysis: Per-realization BER

BER for $\mathbf{e}_\ell$, starting in the ith position, is given by

$$P_{i,\ell}(\mathbf{H},\mathbf{J}) = a_\ell \cdot \mathrm{PEP}_{i,\ell}(\mathbf{H},\mathbf{J})$$

Sum over all $\mathbf{e}_\ell \longrightarrow$ BER for the ith position

$$P_i(\mathbf{H},\mathbf{J}) = \sum_{\ell=1}^{L} P_{i,\ell}(\mathbf{H},\mathbf{J})$$

All positions equally likely $\longrightarrow$ BER $P(\mathbf{H},\mathbf{J})$ is

$$P(\mathbf{H},\mathbf{J}) = \frac{1}{L_c}\sum_{i=1}^{L_c} \min\left[\frac{1}{2}, \sum_{\ell=1}^{L} P_{i,\ell}(\mathbf{H},\mathbf{J})\right]$$

This is the (approximate) BER for a given **H** and **J**

Analysis: Average and Outage BER

Once we know $P(\mathbf{H}, \mathbf{J})$, we can easily obtain both average and outage BER

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Average BER

$$P_{\mathrm{avg}} = \frac{1}{N_c} \sum_{n=1}^{N_c} \mathbb{E}_{\mathbf{J}} \left\{ P(\mathbf{H}_n, \mathbf{J}) \right\}$$

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Outage BER

$$P_{\mathrm{out}} = \max_{\mathbf{H}_n \in \mathcal{H}_{\mathrm{in}}} \mathbb{E}_{\mathbf{J}} \left\{ P(\mathbf{H}_n, \mathbf{J}) \right\}$$

(where $\mathcal{H}_{\mathrm{in}}$ are the channels that are NOT in outage)

Analysis: Direct Average BER

A Special Case

If channel gains **h** are

- Rayleigh-distributed r.v.s, and
- Correlated with correlation matrix $\boldsymbol{\Sigma}_{\mathbf{hh}}$

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$$\mathrm{PEP}_{i,\ell} = \Pr\left\{||\mathbf{r} - \mathbf{Hx}||^2 > ||\mathbf{r} - \mathbf{Hz}_{i,\ell}||^2 \;\middle|\; \mathbf{H}, \mathbf{J}\right\}$$

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We only need to consider non-zero entries of $(\mathbf{x} - \mathbf{z}_{i,\ell}) \longrightarrow$ make

- $\mathbf{x}'$, $\mathbf{z}'_{i,\ell}$, $\mathbf{H}' = \mathrm{diag}(\mathbf{h}')$, $\mathbf{J}'$ and $\mathbf{n}'$
- $\boldsymbol{\Sigma}_{\mathbf{h}'\mathbf{h}'}$
- $\mathbf{D} = \mathrm{diag}(\mathbf{x}' - \mathbf{z}'_{i,\ell})$
- $\mathbf{g} = \mathbf{H}'(\mathbf{x}' - \mathbf{z}'_{i,\ell}) = \mathbf{Dh}'$

Analysis: Direct Average BER

We have

$$\begin{aligned} \mathbb{E}(\mathbf{g}) &= \mathbf{0}_{\eta\times 1} \\ \mathbb{E}(\mathbf{g}\mathbf{g}^H) &= \mathbf{R}_{\mathbf{gg}} = \mathbf{D}\boldsymbol{\Sigma}_{\mathbf{h}'\mathbf{h}'}\mathbf{D}^H \end{aligned}$$

i.e., the distribution of $\mathbf{g}$ is zero-mean complex Gaussian with covariance matrix $\mathbf{R}_{\mathbf{gg}}$.

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Rewriting the PEP to include only contributing terms, we obtain

$$\begin{aligned}\overline{\mathrm{PEP}}_{i,\ell} &= \Pr\left\{||\mathbf{r}' - \mathbf{H}'\mathbf{z}'_{i,\ell}||^2 - ||\mathbf{r}' - \mathbf{H}'\mathbf{x}'||^2 < 0\right\}\\ &= \Pr\left\{\mathbf{g}\mathbf{g}^H - \mathbf{g}(\mathbf{J}' + \mathbf{n}')^H - (\mathbf{J}' + \mathbf{n}')\mathbf{g}^H < 0\right\}\\ &= \Pr\left\{\Delta_{i,\ell}(\mathbf{D}) < 0\right\}\end{aligned}$$

Analysis: Direct Average BER

$\Delta_{i,\ell}(\mathbf{D}) = \mathbf{y}^H \mathbf{A} \mathbf{y}$ and

$$\mathbf{y} = \begin{bmatrix} \mathbf{g} \\ \mathbf{J}' + \mathbf{n}' \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \mathbf{I}_\eta & -\mathbf{I}_\eta \\ -\mathbf{I}_\eta & \mathbf{0}_\eta \end{bmatrix}$$

and $\mathbf{y}$ has

- mean $\boldsymbol{\mu}_{\mathbf{yy}}$
- covariance matrix $\mathbf{R}_{\mathbf{yy}}$

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Laplace transform approach to get $\Pr\left\{\Delta_{i,\ell}(\mathbf{D}) < 0\right\}$

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non-faded interferers

$$\Phi_{i,\ell}(s) = \frac{\exp[-s\boldsymbol{\mu}_{\mathbf{yy}}^{H}(\mathbf{A}^{-1} + s\mathbf{R}_{\mathbf{yy}})^{-1}\boldsymbol{\mu}_{\mathbf{yy}}]}{\det(\mathbf{I}_{2\eta} + s\mathbf{R}_{\mathbf{yy}}\mathbf{A})}$$

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iid Rayleigh interferers

$$\Phi_{i,\ell}(s) = \frac{1}{\det(\mathbf{I}_{2\eta} + s\mathbf{R}_{\mathbf{yy}}\mathbf{A})}$$

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iid Rayleigh interferers

$$\Phi_{i,\ell}(s) = \frac{1}{\det(\mathbf{I}_{2\eta} + s\mathbf{R}_{\mathbf{yy}}\mathbf{A})}$$

and

$$\overline{\mathrm{PEP}}_{i,\ell} = \Pr\left\{\Delta_{i,\ell}(\mathbf{D}) < 0\right\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \Phi_{i,\ell}(s)\frac{\mathrm{d}s}{s}$$

Analysis: Direct Average BER

$\overline{\mathrm{BER}}$ for $\mathbf{e}_\ell$, starting in the ith position, is given by

$$\bar{P}_{i,\ell} = a_\ell \cdot \overline{\mathrm{PEP}}_{i,\ell}$$

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All positions equally likely $\longrightarrow \overline{\mathrm{BER}}$ is

$$\bar{P} = \frac{1}{L_c} \sum_{i=1}^{L_c} \bar{P}_i = \frac{1}{L_c} \sum_{i=1}^{L_c} \sum_{\ell=1}^{L} \bar{P}_{i,\ell}$$

This is the (approximate) average BER

Results: MB-OFDM System and Channel Models

We will consider Multiband OFDM as an example OFDM system

MB-OFDM

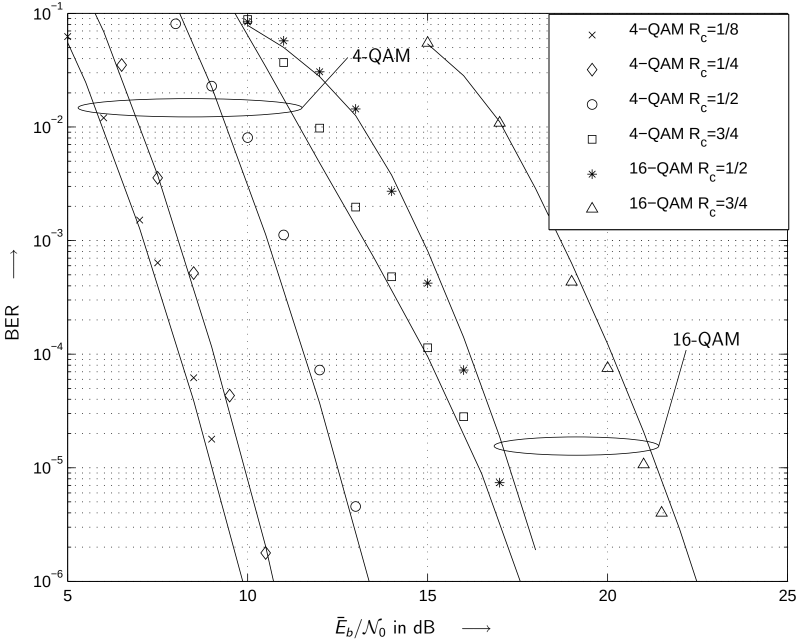
- Multiband OFDM is a leading high rate UWB system
- 3.1–10.6 GHz $\longrightarrow$ 14 subbands of 528 MHz
- QPSK modulation (also consider 16-QAM)
- Punctured convolutional code

Channel impulse response (802.15.3a model)

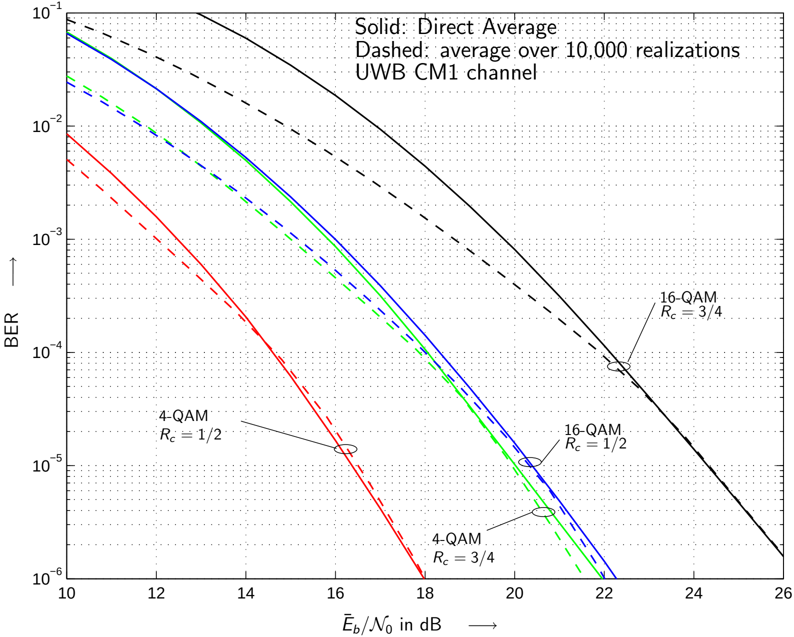
$$h(t) = X \sum_{l \geq 0} \sum_{k \geq 0} \alpha_{k,l} \delta(t - T_l - \tau_{k,l})$$

- Impulse response consists of clusters of multipath components
- We will consider equivalent frequency-domain channel $\mathbf{H}$
- Channel is quasi-static
- For given X, gains $\mathbf{H}$ are complex Gaussian

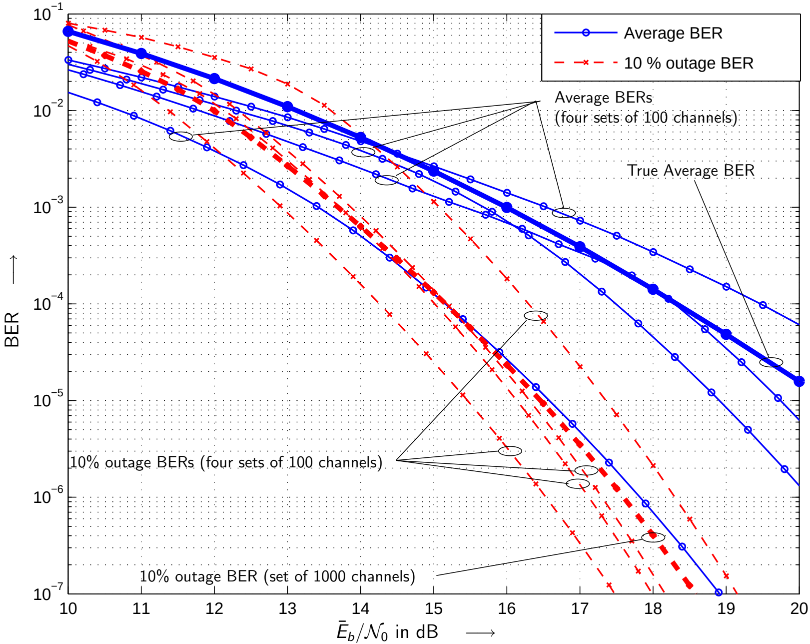
Results: No Interference, 10% Outage BER



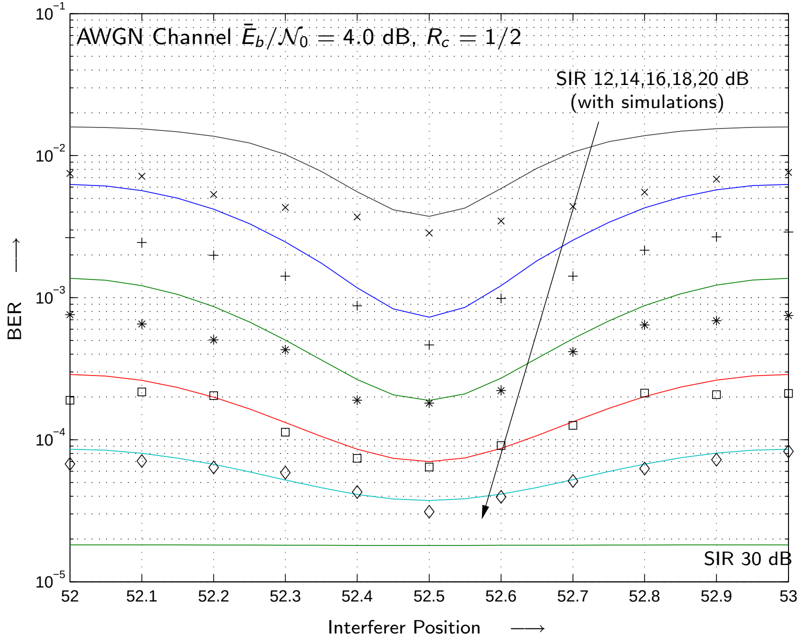
Results: No Interference, Average BER



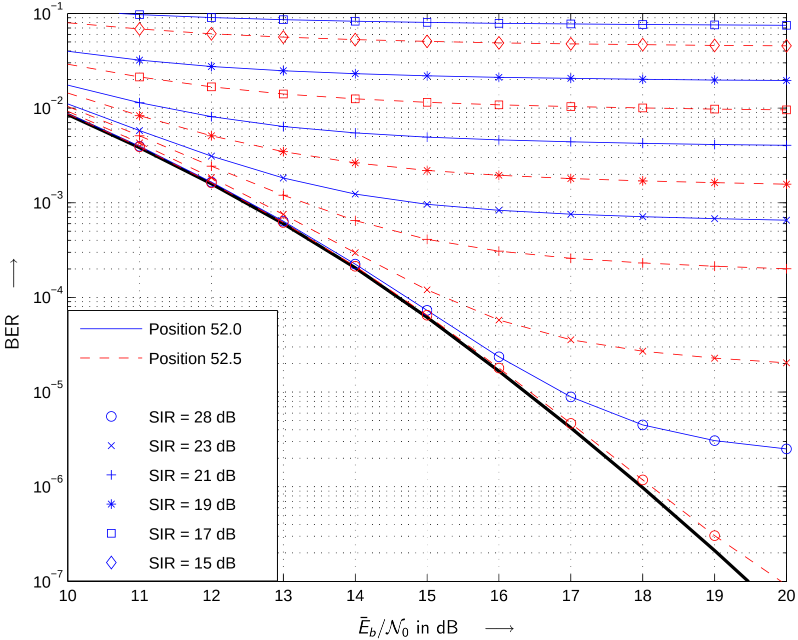
Results: No Interference, Different Channel Realizations



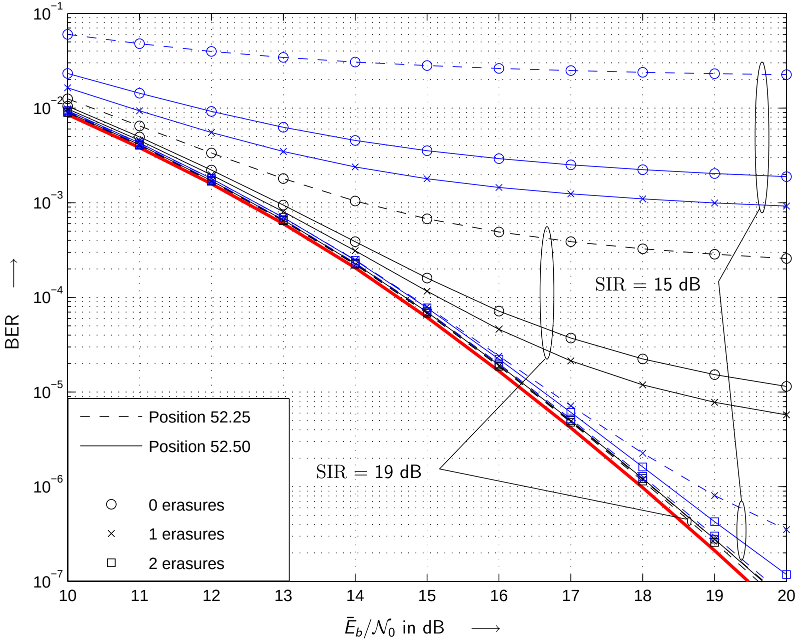
Results: Interference, AWGN Channel



Results: Rayleigh Interference, Direct Average BER



Results: Interference, Average BER w/ Erasures



Conclusions

Analysis

Developed methods to estimate

- Outage BER
- Average BER

These methods apply to any OFDM system in quasi-static channel

Results

- Can approximate OFDM performance without simulations
- Both average and outage BER easily available
- Small set of channel realizations ⟶ misleading results