Error Rate Analysis for Coded Multicarrier Systems over Quasi-static Fading Channels

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Outline

Overview of Concepts

Orthogonal Frequency Division Multiplexing (OFDM)

- Many orthogonal subchannels via IFFT & FFT
- (Punctured) convolutional codes for error protection
- Robust in frequency-selective fading

Quasi-Static Fading Channel

- Fading conditions constant for “relatively long” period of time
- Here: “relatively long” = (at least) one packet
- Can only code over one channel realization
- Channel is also frequency-selective

Analysis: Performance Measures

We will consider two different performance measures

Average Bit Error Rate (BER)

Average BER of a (large) number of different channel realizations

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Outage BER

- if we assume $X\%$ of channels are bad ("in outage"), what is the worst-case performance of the remaining $(100 - X)\%$ of the channels?
- (we will consider $X = 10\%$ outage probability)

Outage BER is a more relevant measure for quasi-static channels...
BUT, it is hard to determine.

Analysis: BER for Quasi-static Channels

Average BER for Fast Fading Channel

- Get pairwise error probability $P_2(d|h)$ for given h
- Integrate over pdf of h to get $P_2(d)$, then apply union bound

$$\mathrm{BER} \leq \frac{1}{k} \sum_{d=d_{\mathrm{free}}}^{\infty} \beta_d P_2(d)$$

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Quasi-static Channel

Each coded block transmitted over one channel realization

- Only "sees" a limited number of channel gains
- (and, correlation between bits depends on starting position)

CANNOT simply integrate over pdf of fading distribution!

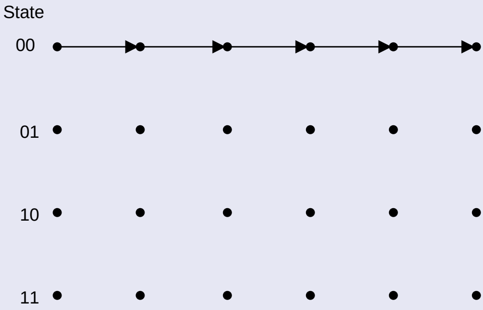
Question

Can we predict the BER of a given realization?

Analysis: Error Vectors

Trellis representation of $R_c = 1/2$ (7,5) convolutional code

An example error path



Input: $\{\ ,\ ,\ ,\ \}$

Output $\mathbf{e}_1$: $\{\ ,\ ,\ ,\ ,\ ,\ ,\ ,\ \}$

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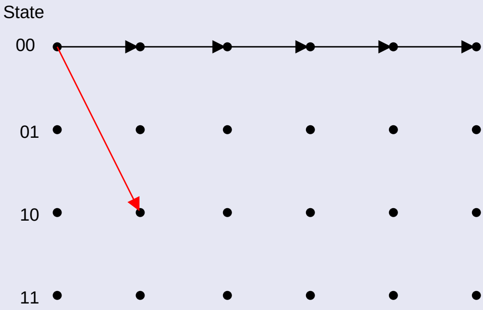

Input: $\{1, \ , \ , \ \}$

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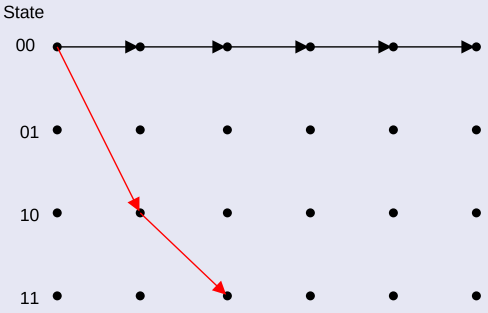

Input: $\{1, 1, \ , \ \}$

Output $\mathbf{e}_1$: $\{1, 1, 0, 1, \ , \ , \ , \ \}$

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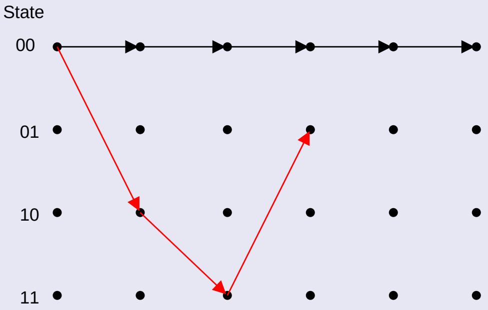

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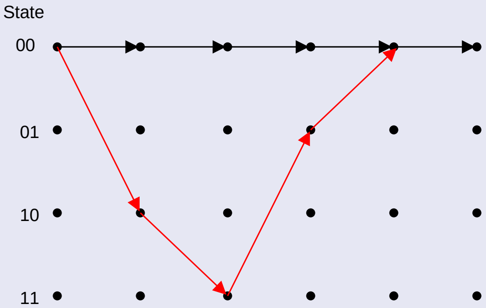

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- Call this set $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_L\}$
- l_ℓ is the length of $\mathbf{e}_\ell$
- a_ℓ is the number of information bit errors associated with $\mathbf{e}_\ell$

$\mathcal{E}$ contains all the "most likely" error events (the ones with shortest length)

Analysis: System Model

Frequency-domain System Model

$$\mathbf{r} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

where

- $\mathbf{x}$ are the transmitted symbols
- $\mathbf{H} = \mathrm{diag}(\mathbf{h})$ is a (diagonal) matrix of channel gains
- $\mathbf{n}$ are AWGN noise variables

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- For each error vector $\mathbf{e}_\ell$, we form the "full" error codeword (length L_c)

$$\mathbf{q}_{i,\ell} = [\underbrace{0 \;\; 0 \;\; \ldots \;\; 0}_{i-1} \;\; \underbrace{\mathbf{e}_\ell}_{l_\ell} \;\; \underbrace{0 \;\; 0 \;\; \ldots \;\; 0}_{L_c - l_\ell - i + 1}]^T$$

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- Now, given $\mathbf{q}_{i,\ell}$, the competing codeword is given by

$$\mathbf{v}_{i,\ell} = \mathbf{c} \oplus \mathbf{q}_{i,\ell}$$

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- the PEP for the ℓth error vector starting in the ith position, i.e., the probability that $\mathbf{v}_{i,\ell}$ is detected given that $\mathbf{c}$ was transmitted, is given by

$$\mathrm{PEP}_{i,\ell}(\mathbf{H}) = \Pr\left\{ ||\mathbf{r} - \mathbf{H}\mathbf{x}||^2 > ||\mathbf{r} - \mathbf{H}\mathbf{z}_{i,\ell}||^2 \;\middle|\; \mathbf{H} \right\}$$

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- For given $\mathbf{H}, i, \ell$

PEP

$$\mathrm{PEP}_{i,\ell}(\mathbf{H}) = Q\left(\sqrt{\frac{||\mathbf{H}(\mathbf{x} - \mathbf{z}_{i,\ell})||^2}{2\mathcal{N}_0}} \right)$$

Analysis: Per-realization BER

BER for $\mathbf{e}_\ell$, starting in the ith position, is given by

$$P_{i,\ell}(\mathbf{H}) = a_\ell \cdot \mathrm{PEP}_{i,\ell}(\mathbf{H})$$

Sum over all $\mathbf{e}_\ell \longrightarrow$ BER for the ith position

$$P_i(\mathbf{H}) = \sum_{\ell=1}^{L} P_{i,\ell}(\mathbf{H})$$

All positions equally likely $\longrightarrow$ BER $P(\mathbf{H})$ is

$$P(\mathbf{H}) = \frac{1}{L_c} \sum_{i=1}^{L_c} \min\left[\frac{1}{2}, \sum_{\ell=1}^{L} P_{i,\ell}(\mathbf{H})\right]$$

This is the (approximate) BER for a given $\mathbf{H}$

Analysis: Average and Outage BER

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Average BER

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Average BER

$$P_{\text{avg}} = \frac{1}{N_c} \sum_{j=1}^{N_c} P(\mathbf{H}_j)$$

Outage BER

$$P_{\text{out}} = \max_{\mathbf{H}_j \in \mathcal{H}_{\text{in}}} P(\mathbf{H}_j)$$

(where $\mathcal{H}_{\text{in}}$ are the channels that are NOT in outage)

Analysis: Direct Average BER

A Special Case

If channel gains $\mathbf{h}$ are

- Rayleigh-distributed r.v.s, and
- Correlated with correlation matrix $\boldsymbol{\Sigma}_{\mathbf{hh}}$

we can obtain the average BER directly

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Start from PEP for position i, error vector ℓ

$$\mathrm{PEP}_{i,\ell}(\mathbf{H}) = \Pr\left\{ ||\mathbf{r} - \mathbf{Hx}||^2 > ||\mathbf{r} - \mathbf{Hz}_{i,\ell}||^2 \;\middle|\; \mathbf{H} \right\}$$

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We only need to consider non-zero entries of $(\mathbf{x} - \mathbf{z}_{i,\ell}) \longrightarrow$ make

- $\mathbf{x}'$, $\mathbf{z}'_{i,\ell}$, $\mathbf{H}' = \mathrm{diag}(\mathbf{h}')$, and $\mathbf{n}'$
- $\boldsymbol{\Sigma}_{\mathbf{h}'\mathbf{h}'}$
- $\mathbf{D} = \mathrm{diag}(\mathbf{x}' - \mathbf{z}'_{i,\ell})$
- $\mathbf{g} = \mathbf{H}'(\mathbf{x}' - \mathbf{z}'_{i,\ell}) = \mathbf{D}\mathbf{h}'$

Analysis: Direct Average BER

$$\begin{aligned} \mathbb{E}(\mathbf{g}) &= \mathbf{0}_{\eta\times 1} \\ \mathbb{E}(\mathbf{g}\mathbf{g}^H) &= \mathbf{R}_{\mathbf{gg}} = \mathbf{D}\boldsymbol{\Sigma}_{\mathbf{h}'\mathbf{h}'}\mathbf{D}^H \end{aligned}$$

i.e., the distribution of $\mathbf{g}$ is zero-mean complex Gaussian with covariance matrix $\mathbf{R}_{\mathbf{gg}}$.

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i.e., the distribution of $\mathbf{g}$ is zero-mean complex Gaussian with covariance matrix $\mathbf{R_{gg}}$. Following [Veervalli '01], we can write the bit error probability for the j^{th} error vector starting in the i^{th} position as

$$\bar{P}_{i,\ell} = \frac{a_\ell}{\pi}\int_0^{\pi/2}\left[\det\left(\frac{E_s\mathbf{R_{gg}}}{\mathcal{N}_0\sin^2\theta} + \mathbf{I}\right)\right]^{-1} d\theta$$

Sum over ℓ and average over i as before $\longrightarrow \overline{\mathrm{BER}}$ is

$$\bar{P} = \frac{1}{L_c}\sum_{i=1}^{L_c}\bar{P}_i = \frac{1}{L_c}\sum_{i=1}^{L_c}\sum_{\ell=1}^{L}\bar{P}_{i,\ell}$$

This is the (approximate) average BER

Results: MB-OFDM System and Channel Models

We will consider Multiband OFDM as an example OFDM system

MB-OFDM

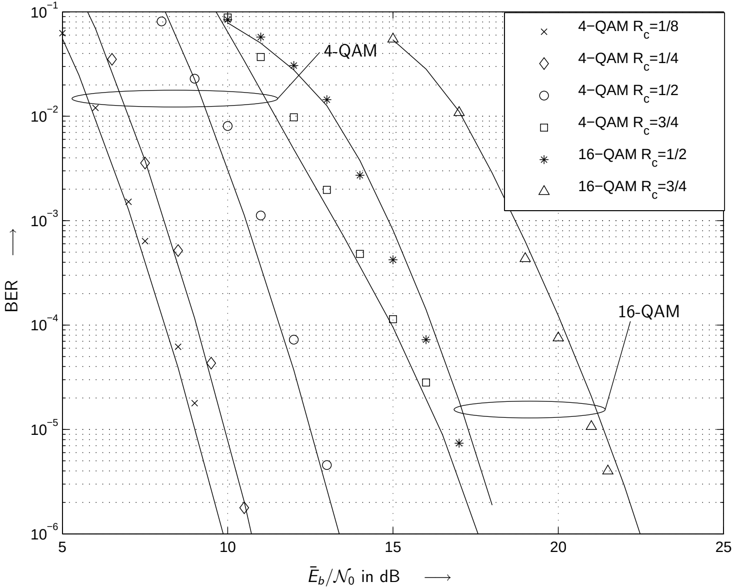
- Multiband OFDM is a leading high rate UWB system
- 3.1–10.6 GHz $\longrightarrow$ 14 subbands of 528 MHz
- QPSK modulation (also consider 16-QAM)
- Punctured convolutional code

Channel impulse response (802.15.3a model)

$$h(t) = X \sum_{l \geq 0} \sum_{k \geq 0} \alpha_{k,l} \delta(t - T_l - \tau_{k,l})$$

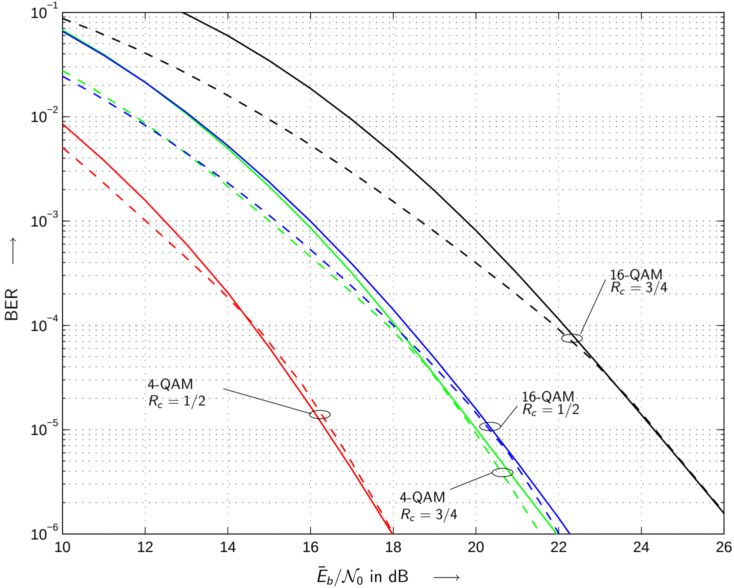
- Impulse response consists of clusters of multipath components
- We will consider equivalent frequency-domain channel $\mathbf{H}$
- Channel is quasi-static
- For given X, gains $\mathbf{H}$ are complex Gaussian

Results: Outage BER



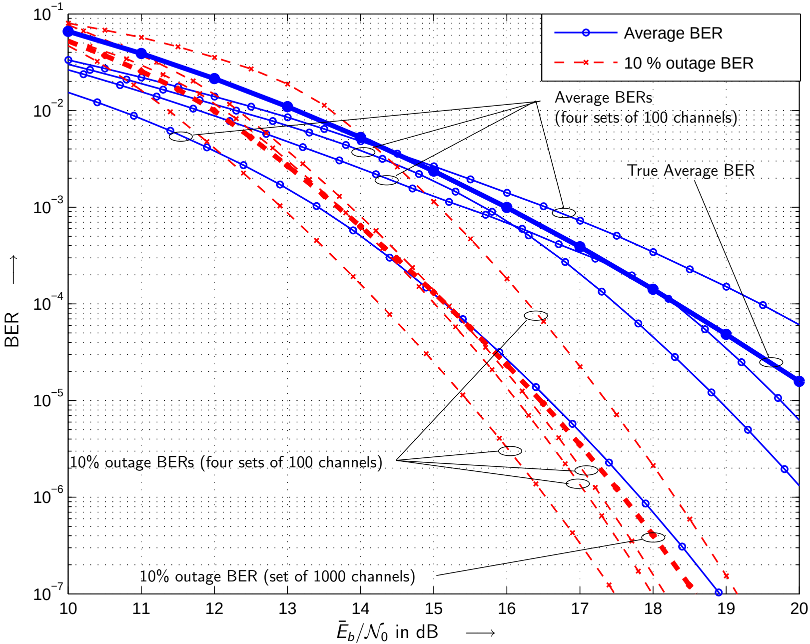
10% Outage BER. 100 CM1 channel realizations.

Results: Average BER



Solid: Direct average. Dashed: average over 10,000 channel realizations.

Results: Different Channel Realizations



Conclusions

Analysis

Developed methods to estimate

- Outage BER
- Average BER

These methods apply to any OFDM system in quasi-static channel

Results

- Can approximate OFDM performance without simulations
- Both average and outage BER easily available
- Small set of channel realizations $\longrightarrow$ misleading results

More detailed journal preprint at www.ece.ubc.ca/∼csnow/